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ON DIRECTIONAL DERIVATIVES OF THE THETA FUNCTION ALONG ITS DIVISOR

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ABSTRACT

In this note we will identify the divisor of any fixed directional derivative of the Riemann theta function evaluated along the theta divisor with the divisor formed from a $g-1$ dimensional subspace of holomorphic one-forms from the underlying Riemann surface.

Introduction

The classical Riemann vanishing theorem can be viewed as identifying the divisor of the theta function $\theta(z,\Omega)$ associated to a compact Riemann surface $\mathbf{X}$ with the divisor of the determinant of holomorphic one-forms from $\mathbf{X}$, evaluated on $\mathbf{X}^g$. In [F] Farkas shows that any partial derivative $\frac{\partial\theta}{\partial z_j}(z,\Omega)$ of the theta function when evaluated at $z=\mathcal{K}_P$, the vector of Riemann constants corresponding to the base point P, does not vanish identically in P. Further, it is shown that the divisor of $\frac{\partial\theta}{\partial z_j}(\mathcal{K}_P,\Omega)$ coincides with the divisor of a certain Wronskian corresponding to a $g-1$ dimensional subspace of holomorphic one-forms.

In this note we will extend Farkas's theorem by identifying the zero set of any directional derivative of the theta function evaluated at any point in the theta divisor with the divisor of a determinant formed from a $g-1$ dimensional subspace of holomorphic one-forms.

Riemann's theorem is proved by studying certain integrals involving $\theta(z,\Omega)$ restricted to an image of $\mathbf{X}$ in its Jacobian variety $J(\mathbf{X})$, with the integration being over a canonical dissection of $\dot{\mathbf{X}}$. This proof can be found in many texts, such as [Fal], [F-K], or [Mu]. Essentially, the proof is an application of the

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argument principle. Farkas modifies this proof by studying the integrand given by restricting the theta function to n times the image of $\mathbf{X}$ in $J(\mathbf{X})$, for any integer n. Our result comes from studying the limiting behavior of analytic torsion, a function defined on $J(\mathbf{X})$ when the Jacobian is viewed as the moduli space of degree zero line bundles on $\mathbf{X}$. For completeness, we shall present a review of the problem of analytic torsion for degree zero line bundles on Riemann surfaces and state the main result derived in [J], which is a precise evaluation of analytic torsion in terms of theta functions and related quantities. It is through the study of this evaluation that we are able to derive the result presented in this paper.

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1. Preliminary Results

Let $\mathbf{X}$ denote a compact Riemann surface of positive genus g, marked by a canonical basis of its first homology group $H_1(\mathbf{X},\mathbf{Z})$, which we write as $\{A_1\dots B_g\}$. Dual to this marking is a basis of holomorphic one-forms $\{\zeta_j\}_1^g$, and a corresponding period matrix Ω formed by defining $\Omega_{ij}=\int_{B_j}\zeta_i$. Let $J(\mathbf{X})$ be the Jacobian variety of $\mathbf{X}$, realized as the g-complex dimensional algebraic torus $\mathbf{C}^g$ modulo the lattice $L(\mathbf{X})$ generated by $\mathbf{Z}^g$ and $\Omega\cdot\mathbf{Z}^g$. The Abel-Jacobi map from the space of divisors on $\mathbf{X}$ to $J(\mathbf{X})$ will be denoted as ψ_P where P is the base point.

The theta function with characteristics $\alpha,\beta\in\mathbf{R}^g$ is the entire function on $\mathbf{C}^g$ defined by the series

$$(1)\qquad \theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\Omega)=\sum_{n\in\mathbf{Z}^g}\exp(\pi i^t(n+\alpha)\Omega(n+\alpha)+2\pi i^t(n+\alpha)(z+\beta)).$$

If both α and β are zero we shall write (1) as $\theta(z,\Omega)$. Basic properties of this remarkable function are given in the references [F-K], [Fa1] and [Mu]. For completeness, let us briefly review some main results we shall need. If $n,m\in\mathbf{Z}^g$ then

$$(2)\qquad \theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z+m+\Omega n,\Omega)$$
$$=\exp(-\pi i^t n\Omega n-2\pi i^t nz+2\pi i(^t m\alpha-{}^t n\beta))\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\Omega).$$

A direct calculation verifies that

$$\exp(-2\pi {}^t y Y^{-1} y)\left|\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\Omega)\right|^2, \tag{3}$$

where $y = \mathrm{Im}(z)$ and $Y = \mathrm{Im}(\Omega)$, is a well-defined function on $J(\mathbf{X})$. In other words, (2) is a holomorphic section of a degree zero line bundle over $J(\mathbf{X})$, and (3) represents the square of the norm of this section.

Even though (1) is not well-defined on $J(\mathbf{X})$, by (2) we see that the theta function $\theta(z,\Omega)$ has a well-defined divisor, which we denote by Θ. By Riemann's vanishing theorem ([F-K], page 298), there is a distinguished divisor class $\mathcal{K}$ of degree $g-1$ such that $\theta(\xi,\Omega) = 0$ if and only if

$$\xi = \sum_1^{g-1} \int_P^{P_j} \zeta + \mathcal{K}_P \mod L(\mathbf{X}) \tag{4}$$

for points $P_1 \dots P_{g-1}$ in $\mathbf{X}$. Further, the order of vanishing of $\theta(z,\Omega)$ at $z = \xi$ is equal to $\dim H^0(\mathbf{X}, K \otimes \mathcal{O}(-P_1 \cdots - P_{g-1}))$, where K denotes the canonical bundle on $\mathbf{X}$. In other words, the order of vanishing equals the dimension of the space of holomorphic one-forms with zeros at $P_1, \dots, P_{g-1}$. For generic $P_1 \dots P_{g-1}$, this space has dimension one. The point $\mathcal{K}_P$ in $J(\mathbf{X})$ is called the vector of Riemann constants with respect to the base point P.

Let χ be a unitary character of $H_1(\mathbf{X},\mathbf{Z})$. We shall use $J(\mathbf{X})$ to parameterize the space of unitary characters of $H_1(\mathbf{X},\mathbf{Z})$ by the map ϕ, which we define by

$$u = \phi(\chi) = -\Omega\alpha + \beta, \tag{5}$$

where

$$\alpha_j = \frac{1}{2\pi i}\log\chi(A_j), \qquad \beta_j = \frac{1}{2\pi i}\log\chi(B_j). \tag{6}$$

Throughout we shall use $u = \phi(\chi)$. The degree zero line bundle associated to χ shall be written as $\mathcal{L}_u$. An element of $H^0(\mathbf{X}, K \otimes \mathcal{L}_u)$ is a Prym differential, a holomorphic one-form on the universal cover of $\mathbf{X}$ with multiplicative behavior (6) in the group Γ which uniformizes $\mathbf{X}$ in its universal cover (see [F-K], page 119).

2. Statement of the Main Result

By the Riemann-Roch theorem ([F-K], page 127), $\dim H^0(\mathbf{X}, K \otimes \mathcal{L}_u) = g-1$ if $u \neq 0$ and $\dim H^0(\mathbf{X}, K) = g$ since $u = 0$. Assume that u is not zero and let $\{\eta_u^i\}_1^{g-1}$ be a basis of $H^0(\mathbf{X}, K \otimes \mathcal{L}_u)$. In [J] the following theorem is proved.

THEOREM: *Let $P, P_1, \ldots, P_{g-1}$ be generic points on* $\mathbf{X}$ *and set $Q_1 = P$, $Q_j = P_{j-1}$. Let $\xi = -\Omega\alpha_0 + \beta_0$ be as in (4) and u be as defined in (5). Then, the function $F(u)$ defined by*

$$(7)\quad F(u) = \frac{det(\langle \eta_u^i, \eta_u^j \rangle)}{|det(\eta_u^i(P_j))|^2} \cdot \frac{|det(\zeta_i(Q_j))|^2}{det(Y)} \cdot \frac{\left|\theta\begin{bmatrix} -\alpha(u)+\alpha_0 \\ \beta(u)-\beta_0 \end{bmatrix}(0,\Omega)\right|^2}{\left|\exp(-\pi {}^t\alpha_0 Y \alpha_0)\sum_1^g \frac{\partial\theta}{\partial z_j}(\xi)\zeta_j(P)\right|^2}$$

is independent of the points P and P_j. The function $F(u)$ is non-zero on $J(\mathbf{X})\backslash\{0\}$ with a second order zero at $u = 0$. Further,

$$\frac{\partial^2 F}{\partial u_i^2}(0) = \frac{\partial^2 F}{\partial \bar{u}_j^2}(0) = 0$$

and

$$(8)\qquad \frac{\partial^2 F}{\partial u_i \partial \bar{u}_j} = (Y^{-1})_{ij}.$$

Another way to summarize (8) is to say that near $u = 0$,

$$F(u) = \frac{1}{2}\, {}^t u Y^{-1} \bar{u} + o(|u|^2).$$

Although we do not use this fact, we note that in [J] the function $F(u)$ is the analytic torsion on $\mathbf{X}$ relative to any metric μ on $\mathbf{X}$. For the sake of completeness, let us define the significance of (7) and (8).

Let μ denote a positive $(1,1)$ form on $\mathbf{X}$ which gives a metric on $\mathbf{X}$. Define real operators

$$d = \partial + \bar{\partial} \qquad \text{and} \qquad d^* = \frac{1}{2i}\left(\frac{\partial - \bar{\partial}}{2}\right),$$

so that

$$d^* d = \frac{-1}{2i}\partial\bar{\partial}.$$

Relative to the metric μ on $\mathbf{X}$, the Laplacian Δ_μ is defined by

$$d^* df = (-\frac{1}{4}\Delta_\mu f)\mu.$$

An eigenfunction of the Laplacian f with eigenvalue λ is a smooth function on $\mathbf{X}$ for which the following equation holds.

$$\mathbf{\Delta}_\mu f = \lambda f.$$

This notion extends to sections f_u of the degree zero line bundles $\mathcal{L}_u$. The Laplacian acts on smooth sections of $\mathcal{L}_u$ with a discrete spectrum, which we shall denote by

$$0 \leq \lambda_0(u) \leq \lambda_1(u) \leq \dots.$$

The spectral zeta function associated to this data is defined by

$$\zeta_\mu(s,u) = \sum_{\lambda_n(u)\neq 0} \lambda_n(u)^{-s}. \tag{9}$$

By Weyl's law, (9) is a convergent holomorphic function of s for $\mathrm{Re}(s) > 1$. In [M-P] it is shown that (9) has a meromorphic extension to the complex plane that is holomorphic at $s = 0$. With this, the analytic torsion on $\mathbf{X}$ relative to the metric μ is defined by

$$\det\mathbf{\Delta}_\mu(u) = \exp(-\zeta'_\mu(0,u)). \tag{10}$$

In [J] the analytic torsion $\det\mathbf{\Delta}_\mu(u)$ is studied as a function on $J(\mathbf{X})$ through the map (5). If $u \neq 0, \lambda_0(u) > 0$ and $\lambda_0(0) = 0$. So, in the case $u = 0$ the sum (9) omits the eigenvalue zero, and one uses the notation $\det^*\mathbf{\Delta}_\mu$ to denote the special value (10).

In [J], equation (7) is derived with the function $F(u)$ given by

$$F(u) = \det\mathbf{\Delta}_\mu(u) \cdot (\frac{\mathrm{vol}_\mu \mathbf{X}}{4\pi^2 \det^*\mathbf{\Delta}_\mu}).$$

Equation (8) is simply the Hessian of $\lambda_0(u)$ at $u = 0$.

In this note, we shall study (7) for a family of line bundles $\mathcal{L}_{td}$ where t is complex and approaches zero and $d \in \mathbf{P}^{g-1}$ is fixed. In other words, we will compute the directional derivative of $F(u)$ at $u = 0$ in the direction d through (8) and through (7). Our main theorem, and title of the paper, is the following result.

THEOREM 1: *For any non-zero $d^1, d^2 \in \mathbf{C}^g$ and points $P_1 \dots P_{g-1}$ on $\mathbf{X}$, the following identity holds.*

$$\frac{\sum_1^g \frac{\partial\theta}{\partial z_j}(\xi) d_j^1}{\sum_1^g \frac{\partial\theta}{\partial z_j}(\xi) d_j^2} = \frac{\det\left[d^1|\zeta(P_1)\dots|\zeta(P_{g-1})\right]}{\det\left[d^2|\zeta(P_1)\dots|\zeta(P_{g-1})\right]}. \tag{11}$$

The point ξ is given by

$$\xi = \sum_1^{g-1} \int_P^{P_j} \zeta + \mathcal{K}_P \mod L(\mathbf{X})$$

and the matrices in (11) have been expressed by indicating each of the g columns.

After presenting our proof of Theorem 1, we shall establish some corollaries of the result and show how this extends the theorem of Farkas [F]. Also, since (11) is trivial for $g = 1$, we assume $g \geq 2$. The formula (11) does appear, without proof, in [Fa2]. As we shall see, and as was noted in [F], one of the key difficulties is the non-vanishing of any directional derivative of the theta function.

3. Proof of the theorem

The key lemma is the following statement.

LEMMA 1: *Let d be a fixed point in $\mathbf{P}^{g-1}$. Then,*

$$\lim_{t\to 0} H^0(\mathbf{X}, K \otimes \mathcal{L}_{td}) = \{\Sigma a_j \zeta_j | \Sigma a_j d_j = 0\} \tag{12}$$

As previously stated, for any non-zero u, $\dim H^0(\mathbf{X}, K \otimes \mathcal{L}_u) = g - 1$ and (in the case when $u = 0$), $\dim H^0(\mathbf{X}, K) = g$. Lemma 1 identifies the codimension one subspace of $H^0(\mathbf{X}, K)$ obtained by taking the limit as t approaches zero of $H^0(K \otimes \mathcal{L}_{td})$ for fixed nonzero $d \in \mathbf{P}^{g-1}$. Note that one can identify the space of codimension one subspaces of $\mathbf{C}^g$ with $\mathbf{P}^{g-1}$. Lemma 1 gives an explicit bijection between this $\mathbf{P}^{g-1}$ and the space of directions d from 0 in $J(\mathbf{X})$, which is also a $\mathbf{P}^{g-1}$. The identification of codimension one subspaces of $H^0(\mathbf{X}, K)$ and directions from 0 in $J(\mathbf{X})$ given in (12) will take place in the proof of Lemma 1. Since such an identification is not canonical, we will let the proof of (12) guide us to the natural bijection given in (12).

Proof: For any $u \in J(\mathbf{X})\backslash\{0\}$, let

$$f_u(x) = \eta_u(x)\exp(-2\pi i^t\alpha \int_P^x \zeta) \tag{13}$$

for any non-zero $\eta_u \in H^0(\mathbf{X}, K \otimes \mathcal{L}_u)$. The form f_u is a section of $K \otimes \tilde{\mathcal{L}}_u$, where $\tilde{\mathcal{L}}_u$ is the bundle of degree zero on $\mathbf{X}$ associated to the quasi-character (i.e. non-unitary) defined by $\tilde{\chi}(A_j) = 1$ and $\tilde{\chi}(B_j) = \exp(2\pi i u_j)$. As such, we can make sense of the integrals

$$\int_{A_j} f_u \tag{14}$$

for any A_j. In other words, for each A_j and $\eta_u \in H^0(\mathbf{X}, \mathcal{L}_u \otimes K)$, (14) is a well-defined number. The reason for this is because the quasi-character $\tilde{\chi}$ is trivial along A_j, so the integral (14) is well-defined. By Riemann's bilinear relations, (14) is zero for all j only if f_u, hence η_u, is the zero section.

For any non-zero u, take any basis of $H^0(\mathbf{X}, K \otimes \mathcal{L}_u)$ and use (13) and (14) to define a map

$$\varphi : J(\mathbf{X})\backslash\{0\} \to G(g-1, g) \tag{15}$$

where $G(g-1,g)$ is the Grassman of $(g-1)$-dimensional subspaces of a g-complex dimensional vector space. As stated above, the space $G(g-1,g)$ is isomorphic, although not canonically, to $\mathbf{P}^{g-1}$. Later, we shall choose an isomorphism, but, for now, let us keep the map (15) as defined without changing the image manifold.

Let U denote a neighborhood of 0 in $J(\mathbf{X})$ such that $U \setminus \{0\}$ is conformally equivalent to $D^* \times \mathbf{P}^{g-1}$, where D^* denotes the punctured disc in $\mathbf{C}$. Choose coordinates t and d where $(t,d) \in D^* \times \mathbf{P}^{g-1}$ which parameterize $U \setminus \{0\}$ in $J(\mathbf{X})$, and let φ_t be the map (15) viewed as a function of $t \in D^*$ with $d \in \mathbf{P}^{g-1}$ being fixed. We shall write $u \in U$ as $u = td$. For any non-zero $u \in J(\mathbf{X})$, one can choose a basis of $H^0(\mathbf{X}, K \otimes \mathcal{L}_u)$ which varies holomorphically in u (see [J], (3.3) for an explicit construction using the prime form). For such a basis, the divisors vary holomorphically in u, hence holomorphically in t for fixed d. Choose a basis $\{\eta_u\}$ that varies holomorphically in u, and consider the family of maps from D^* into $\mathbf{X}$ obtained by composing φ_t with the map which takes the divisor of η_u. Let us use the same symbol φ_t to denote this map. Thus, φ_t can be viewed as a family of maps of D^* into $\mathbf{X}$. Since $g \geq 2$, any such map extends to a map of D into $\mathbf{X}$ (see [La], page 40; this uses the hyperbolicity of $\mathbf{X}$ which is guaranteed because $g \geq 2$). Thus we can extend φ_t for fixed $d \in \mathbf{P}^{g-1}$ to $t = 0$ as follows. By extending the family of maps from D^* into $\mathbf{X}$, we obtain the divisors of holomorphic one-forms in the limit space

$$\lim_{t \to 0} H^0(\mathbf{X}, K \otimes \mathcal{L}_{td}).$$

This, in turn, determines the A-periods of elements of this codimension one subspace of $H^0(\mathbf{X}, K)$. Since any element of $H^0(\mathbf{X}, K)$ is determined by its A-periods, the map φ_0 is well-defined and determines, for fixed d, the limit subspace of holomorphic one-forms defined in the left-hand-side of (12).

In summary, for any direction $d \in \mathbf{P}^{g-1}$ away from zero in $J(\mathbf{X})$, if u approaches zero along the complex line given by d, the space $H^0(\mathbf{X}, K \otimes \mathcal{L}_u)$ converges to a well-defined $g-1$ complex dimensional subspace of the g-complex dimensional space $H^0(\mathbf{X}, K)$, which is determined by limiting values of the A-periods (14).

To finish the proof of Lemma 1, we shall choose coordinates in order to explicitly describe the map

$$\varphi_0 : \mathbf{P}^{g-1} \to G(g-1, g).$$

Realize $G(g-1,g)$ as $\mathbf{P}^{g-1}$ by the following map: if $A \in G(g-1,g)$, define $d \in \mathbf{P}^{g-1}$ by requiring $\sum_1^g a_j d_j = 0$ for all $(a_j) \in A$. The map φ_0 then becomes a map of $\mathbf{P}^{g-1}$ to itself. Riemann's theorem tells us that the Abel-Jacobi map ψ_P from $\mathbf{X}^g$ to $J(\mathbf{X})$ is generically a bijection ([F-K], page 294 and 298). Combining this with the Implicit Function Theorem ([K], page 43) we conclude that for generic $t \in D^*$ φ_t is of degree one, hence so is φ_0. Relative to the above realization of $G(g-1,g)$ as $\mathbf{P}^{g-1}$, Lemma 5.1 of [J] proved that φ_0 acts as the identity for any d of the form $(\zeta_1(P), \ldots, \zeta_g(P))$ for $P \in \mathbf{X}$. By varying P, these vectors form a basis of $\mathbf{C}^g$. Thus, φ_0 is the identity.

This completes the proof of (12). ∎

Let d^1 and d^2 be non-zero points in $\mathbf{C}^g$ and consider

$$\lim_{t \to 0} \frac{F(td^1)}{F(td^2)} \tag{16}$$

where $F(u)$ is defined by (7), and t is complex. Equation (8) implies that the limit (16) exists and equals

$$\frac{{}^t d^1 Y^{-1} \bar{d^1}}{{}^t d^2 Y^{-1} \bar{d^2}}. \tag{17}$$

By (8), we have that the right hand side of (7) vanishes to order two, and that comes from the theta function. In fact,

$$(18)\qquad \lim_{\substack{t\to 0\\ u=td}} \frac{\left|\theta\begin{bmatrix}-\alpha+\alpha_0\\ \beta-\beta_0\end{bmatrix}(0,\Omega)\right|^2}{|u|^2} = \exp(-2\pi{}^t\alpha_0 Y\alpha_0)\frac{\left|\sum_1^g \frac{\partial\theta}{\partial z_j}(\xi)d_j\right|^2}{{}^t d\bar{d}}.$$

It is important to note that (18) (generically) involves first derivatives of the theta function. If $\{\eta_d^j\}$ denotes a basis of (12), with $u = d$, define

$$(19)\qquad g_d(P_1,\ldots,P_{g-1}) = \frac{|\det(\eta_d^i(P_j))|^2}{\det(\langle\eta_d^i,\eta_d^j\rangle)}({}^t dY^{-1}\bar{d}),$$

for fixed (generic) $P_1\ldots P_{g-1}$ in $\mathbf{X}$. Combining (17), (18) and (19), equation (7) becomes

$$(20)\qquad \frac{\left|\sum_1^g \frac{\partial\theta}{\partial z_j}(\xi)d_j^1\right|^2}{\left|\sum_1^g \frac{\partial\theta}{\partial z_j}(\xi)d_j^2\right|^2} = \frac{g_{d^1}(P_1,\ldots,P_{g-1})}{g_{d^2}(P_1,\ldots,P_{g-1})}.$$

It is easy to show that (19) is independent of the basis $\{\eta_d^j\}$ of (12) that has been chosen. Let us choose a specific basis, namely $(\eta_i) = A(\zeta_j)$ where A is the $(g-1)$ by g matrix

$$(21)\qquad \begin{pmatrix} d_2 & -d_1 & 0 & \\ \vdots & & \ddots & \\ d_g & 0 & & -d_1 \end{pmatrix},$$

and $\{\zeta_i\}$ is the basis of $H^0(\mathbf{X},K)$ dual to the given marking on $\mathbf{X}$. With this, in order to simplify (19), we need to evaluate

$$\det(\langle\eta_d^i,\eta_d^j\rangle) = \det({}^t AY\bar{A})$$

and

$$|\det(\eta_d^i(P_j))| = |\det(A(\zeta_i(P_j)))|.$$

These expressions are evaluated by the following Lemmas. Sketches of the proofs are provided. Detailed proofs are given in [J], section 5.

LEMMA 2:

(21) $$\det({}^t A Y \bar{A}) = |d_1|^{2(g-2)}(\det Y)({}^t d Y^{-1}\bar{d})$$

Proof: Let $\hat{A}$ be the matrix with first row equal to ${}^t\bar{d}Y^{-1}$ and row j given by the $(j-1)$st row of A. Then,

$$\det({}^t\bar{\hat{A}}Y\bar{\hat{A}}) = ({}^t\bar{d}Y^{-1}d)\det({}^t AY^{-1}A)$$
$$= |\det\hat{A}|^2\det Y.$$

By expanding by minors along the first row of $\hat{A}$ we obtain

$$\det\hat{A} = (-d_1)^{g-2}({}^t dY^{-1}\bar{d}),$$

which gives the desired result. ∎

LEMMA 3:

(22) $$\det(A(\zeta_i(P_j))) = -d_1^{g-2}det[d|\zeta(P_1)\ldots|\zeta(P_{g-1})].$$

Proof: Expand (22) by minors along the bottom row and use (22) as an induction hypothesis, with the case $g=2$ being trivially true. ∎

Combining (21) and (22) we have

(23) $$g_d(P_1\ldots P_{g-1}) = \left|\det[d|\zeta(P_1)\ldots|\zeta(P_{g-1})]\right|^2(\det Y)^{-1},$$

which completes the proof of Theorem 1.

To summarize, the proof of Theorem 1 is as follows. By (8), the ratio of directional derivatives of $F(u)$ at zero along the directions d^1 and d^2 (16) is given by the ratio of the limit of (7). Specifically, we obtain

(24) $$\frac{{}^t d^1 Y^{-1}\bar{d}^1}{{}^t d^2 Y^{-1}\bar{d}^2} = \frac{|\sum_1^g \frac{\partial\theta}{\partial z_j}(\xi)d_j^1|^2}{|\sum_1^g \frac{\partial\theta}{\partial z_j}(\xi)d_j^2|^2}\cdot\frac{g_{d^2}(P_1,\ldots,P_{g-1})}{g_{d^1}(P_1,\ldots,P_{g-1})}\cdot\frac{{}^t d^1 Y^{-1}\bar{d}^1}{{}^t d^2 Y^{-1}\bar{d}^2}.$$

We have used that the vanishing property of $F(u)$ near $u=0$ as given by (8) implies (24) involves the first derivative of the theta function. The above Lemmas evaluate g_d as given by (23). By dropping the absolute values, we obtain a non-zero holomorphic map of modulus one from $\mathbf{P}^{g-1}$ to $\mathbf{C}$ given by the ratio of the two sides of (11) keeping d^2 fixed and varying d^1. This map must be constant which is easily seen to be one by taking $d^1 = d^2$.

4. Corollaries of the main Theorem

If we let points $P_1 \dots P_{g-1}$ coalesce to a point P we obtain

COROLLARY 1:

$$\frac{\sum_1^g \frac{\partial\theta}{\partial z_j}(\mathcal{K}_P)d_j^1}{\sum_1^g \frac{\partial\theta}{\partial z_j}(\mathcal{K}_P)d_j^2} = \frac{det[d^1|\zeta(P)\dots|\zeta^{(g-2)}(P)]}{det[d^2|\zeta(P)\dots|\zeta^{(g-2)}(P)]} \tag{25}$$

where $\zeta^{(h)}$ denotes the column of h derivatives of the holomorphic one-forms $\zeta_1 \dots \zeta_g$.

Proof: This follows immediately from Theorem 1 except for the fact that the left-hand-side involves the first derivatives of the theta function. For this, note that if points P_j coalesce to a single point, (7) shows that during this process the term

$$\sum_1^g \frac{\partial\theta}{\partial z_j}(\xi)\zeta_j(P)$$

vanishes to order g. If $d^1 = \zeta(P)$ and d^2 is fixed, this vanishing in (24) implies

$$\sum \frac{\partial\theta}{\partial z_j}(\mathcal{K}_P)d_j^2$$

does not vanish identically in P. ∎

COROLLARY 2: $\sum_1^g \frac{\partial\theta}{\partial z_j}(\mathcal{K}_P)d_j$ *vanishes at the Weierstrass points of the codimension one subspace of holomorphic one-forms given by* $\{\sum a_j\zeta_j \mid \sum a_j d_j = 0\}$.

If we take $d^1 = \zeta(P)$ in (11) and let the points coalesce we obtain

COROLLARY 3: *In multi-index notation,*

$$\frac{\sum_{|I|=g} \frac{\partial^g\theta}{\partial z^I}(\mathcal{K}_P)\zeta^I(P)}{\sum \frac{\partial\theta}{\partial z_j}(\mathcal{K}_P)d_j} = \frac{W(\zeta)(P)}{det[d|\zeta(P)\dots|\zeta^{(g-2)}(P)]}$$

where $W(\zeta)(P)$ denotes a Wronskian. Thus, the zeros of

$$\sum_{|I|=g} \frac{\partial^g\theta}{\partial z^I}(\mathcal{K}_p)\zeta^I(P) \tag{26}$$

are the Weierstrass points of $\mathbf{X}$.

Equation (26) and the following assertion are given on page 31 of [Fa1].

In the special case $g = 2$ we have the interesting relation

$$frac\frac{\partial\theta}{\partial z_1}(\mathcal{K}_P)\frac{\partial\theta}{\partial z_2}(\mathcal{K}_P) = -\frac{\zeta_2(P)}{\zeta_1(P)}.$$

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